PENRITH HIGH SCHOOL

2016
HSC TRIAL EXAMINATION

Mathematics Extension 2

General Instructions:

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen Black pen is preferred
- Board-approved calculators may be used
- A Reference Sheet of formula is provided
- In questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Answer all Questions in the writing booklets provided

Total marks–100

SECTION I **Pages 3–7**

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

SECTION II **Pages 9–15**

90 marks

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section

Student Name:________________ **Teacher Name:**________________

This paper MUST NOT be removed from the examination room

Assessor: Mr Ferguson

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the modulus and argument of $-1+i$?

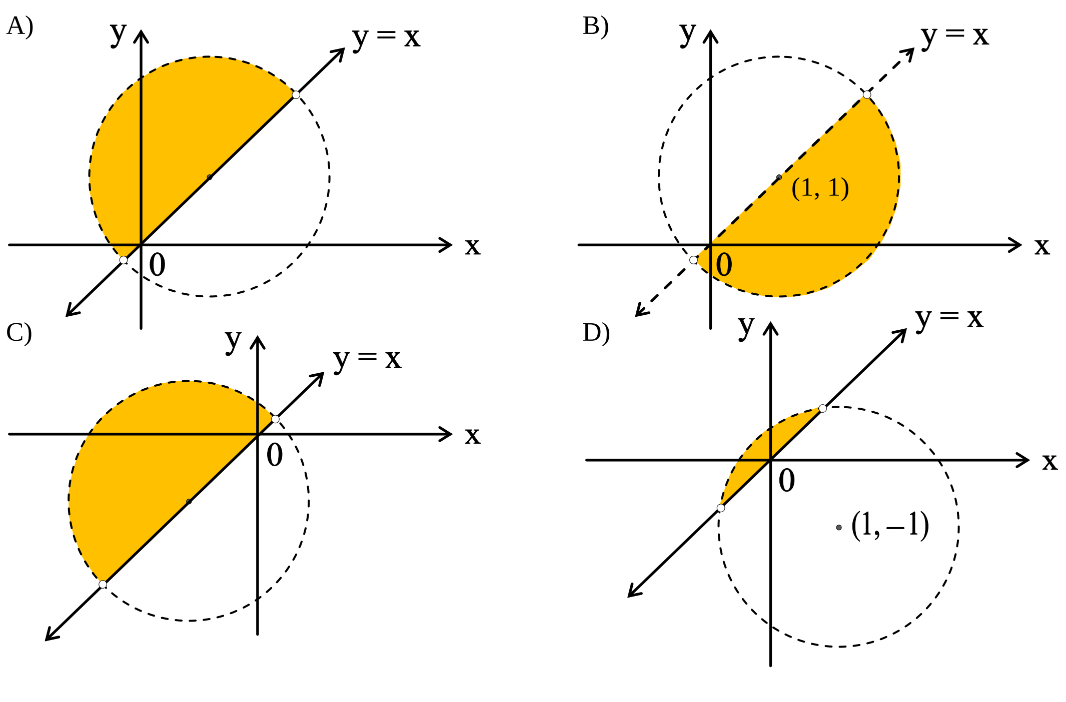
(A) Modulus $\sqrt{2}$ and argument $\dfrac{\pi}{4}$

(B) Modulus $\sqrt{2}$ and argument $\dfrac{3\pi}{4}$

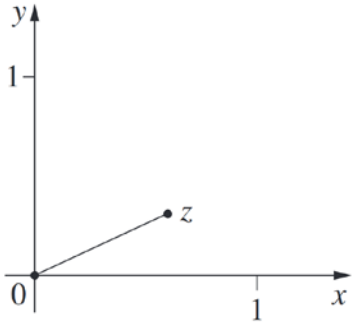
(C) Modulus 2 and argument $\dfrac{\pi}{4}$

(D) Modulus 2 and argument $\dfrac{3\pi}{4}$

2 The complex number z satisfies the inequations $|z-1+i|<2$ and Im (z) $\geq$ Re(z).
Which of the following shows the shaded region in the Argand diagram that satisfies these inequations?

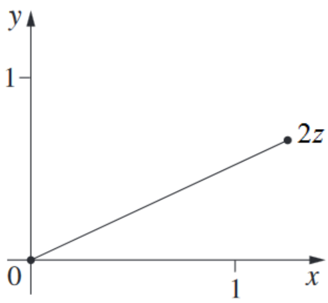


3 The Argand diagram below shows a complex number z.

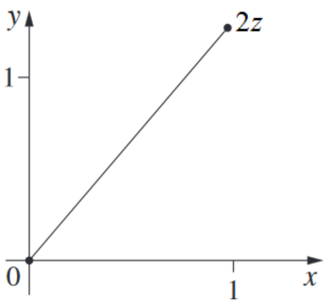


Which diagram best represents $2z$?

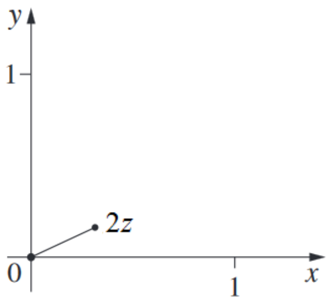
(A)



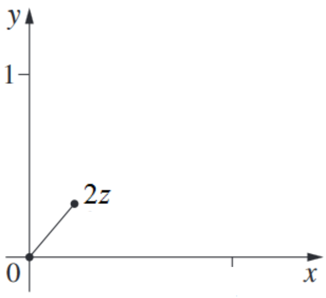
(B)



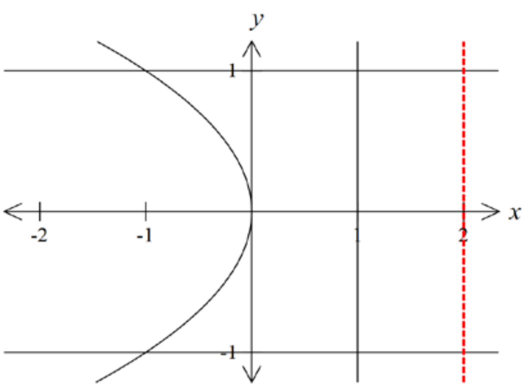
(C)



(D)



4 The region is bounded by the lines $x=1$, $y=1$, $y=-1$ and by the curve $x=-y^2$. The region is rotated through 360° about the line $x=2$ to form a solid. What is the correct expression for volume of this solid?



(A) $V=\int_{-1}^{1}\pi\left(y^4-4y^2+3\right)dy$

(B) $V=\int_{-1}^{1}\pi\left(y^4+4y^2+3\right)dy$

(C) $V=\int_{-1}^{1}\pi\left(y^4-4y^2+4\right)dy$

(D) $V=\int_{-1}^{1}\pi\left(y^4+4y^2+4\right)dy$

5 The foci of the hyperbola $\frac{y^2}{8}-\frac{x^2}{12}=1$ are

(A) $\left(\pm 2\sqrt{5},0\right)$ (B) $\left(\pm\sqrt{30},0\right)$ (C) $\left(0,\pm 2\sqrt{5}\right)$ (D) $\left(0,\pm\sqrt{30}\right)$

6 What are the values of real numbers p and q such that $1+i\sqrt{2}$ is a root of the equation $z^3+pz+q=0$?

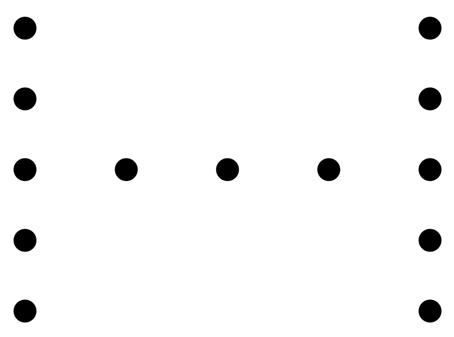
(A) $p=1$ and $q=-6$

(B) $p=-1$ and $q=6$

(C) $p=-1$ and $q=-6$

(D) $p=1$ and $q=6$

7 The diagram shows a shape made by 13 points.



How many triangles can be made with these points as vertices?

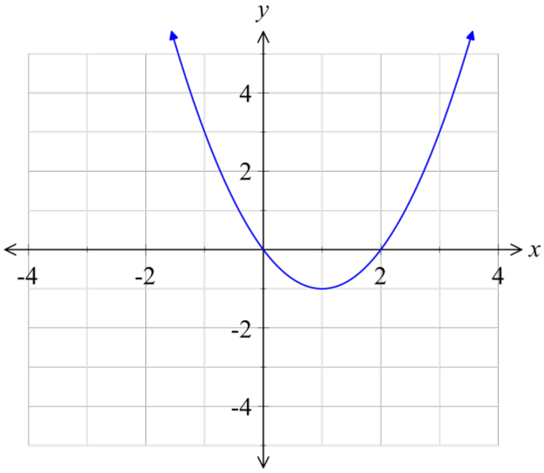
(A) ${}^{13}C_3-3\,{}^5C_3-3$

(B) ${}^{13}C_3-3\,{}^5C_3-4$

(C) ${}^{13}C_3-2\,{}^5C_3-3$

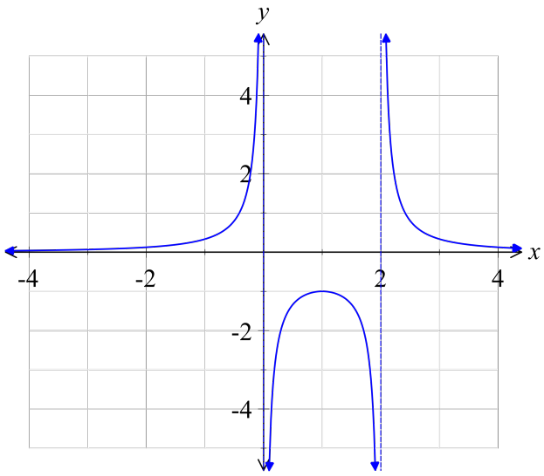
(D) ${}^{13}C_3-2\,{}^5C_3-4$

8 The diagram below shows the graph of the function $y = f(x)$.

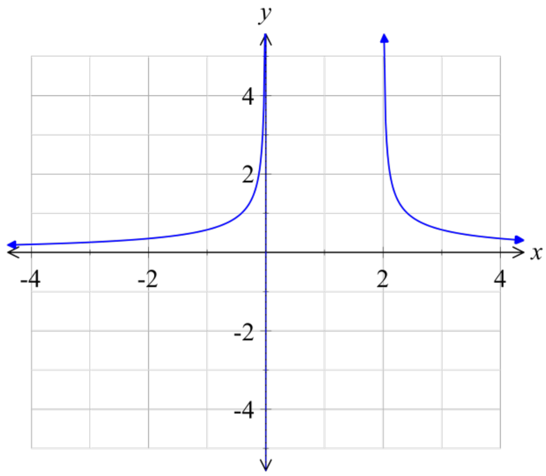


Which of the following is the graph of $y = \dfrac{1}{f(x)}$?

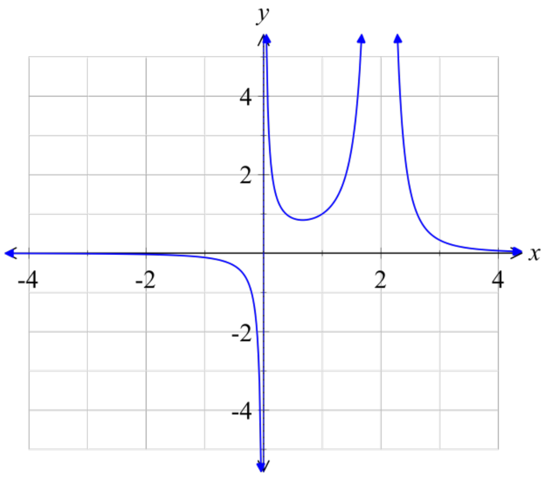
(A)



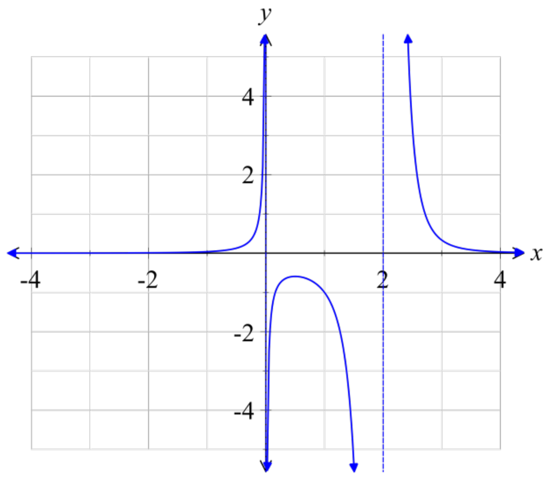
(B)



(C)



(D)



9 Evaluate $\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$

(A) $\frac{\pi}{4}-\frac{1}{2}\ln 2$ (B) $\frac{\pi}{4}+\frac{1}{2}\ln 2$ (C) $\frac{\pi}{4}+2\ln\sqrt{2}$ (D) $\frac{7}{3}$

10 A particle of mass m is moving in a straight line under the action of a force.

$$F=\frac{m}{x^3}(6-10x)$$

Which of the following is an expression for its velocity in any position, if the particle starts from rest at $x=1$?

(A) $v=\pm\frac{1}{x}\sqrt{(-3+10x-7x^2)}$

(B) $v=\pm x\sqrt{(-3+10x-7x^2)}$

(C) $v=\pm\frac{1}{x}\sqrt{2(-3+10x-7x^2)}$

(D) $v=\pm x\sqrt{2(-3+10x-7x^2)}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question on a new writing sheet. Extra writing sheets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing sheet.

a) Let $z = 3 - i$ and $w = 2 + i$. Express the following in the form $x + iy$, where x and y are real numbers:

i) $\dfrac{z}{w}$ **2**

ii) $\overline{-2iz}$ **2**

b) Let $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2} i$.

i) Express z in modulus-argument form. **2**

ii) Show that $z^6 = 1$ **2**

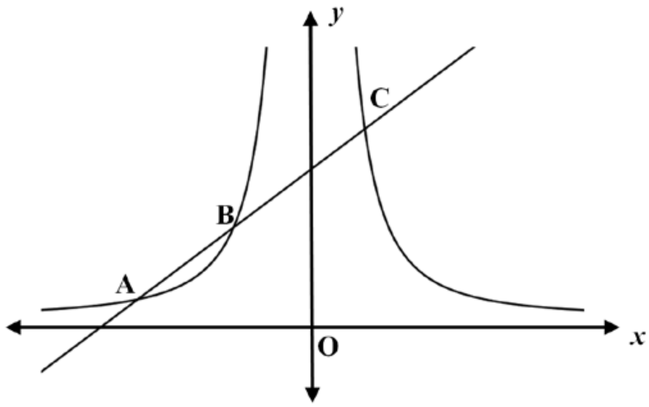
iii) Hence, or otherwise, graph all the roots of $z^6 - 1 = 0$ on an Argand diagram **2**

c) Find $\displaystyle\int \frac{dx}{x\sqrt{x^2 - 1}}$, using $x = \sec\theta$ **2**

d) Show that $\displaystyle\int_0^1 \frac{\sqrt{x}}{1 + x} dx = 2 - \frac{\pi}{2}$ **3**

Question 12 (15 marks) Use a SEPARATE writing sheet.

a)



In the diagram above, the points A, B and C represent the points of intersection of the line $y = 4x + 8$ and the curve $y = \dfrac{1}{x^2}$. The x-values of A, B and C are α, β and γ.

(i) Show that α, β and γ satisfy $4x^3 + 8x^2 - 1 = 0$ **1**

(ii) Find a polynomial with roots α^2, β^2 and γ^2 **2**

(iii) Find $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$ **3**

(iv) Prove that $\text{OA}^2 + \text{OB}^2 + \text{OC}^2 = 132$ where O is the origin **3**

b) If w is a complex root of the equation $z^3 = 1$

(i) Show that $1 + w + w^2 = 0$ **1**

(ii) Find the value of $(1 + 2w + 3w^2)(1 + 2w^2 + 3w)$ **2**

c) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{5 + 3\cos x}\, dx$ **3**

Question 13 (15 marks) Use a SEPARATE writing sheet.

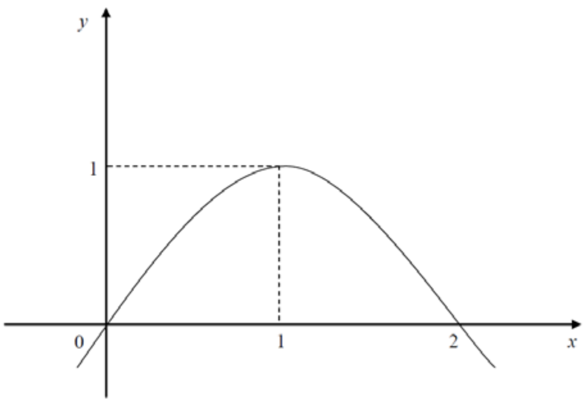
a) By completing the square find $\displaystyle\int \frac{dx}{\sqrt{6x-x^2}}$ **2**

b) (i) Find real constants A, B and C such that

$$\frac{x+4}{x\left(x^2+4\right)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+4}$$ **2**

(ii) Hence find $\displaystyle\int \frac{x+4}{x\left(x^2+4\right)}\,dx$ **2**

c) The graph below shows the curve $y = f(x)$ where $f(x) = x(2-x)$



Without the use of calculus, sketch the following curves. Show any intercepts, asymptotes, end points and turning points. (Use $\frac{1}{3}$ of a page per sketch)

(i) $y = f(2x)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $|y| = f(x)$ **2**

(iv) $y = \ln f(x)$ **2**

(v) $y = f(e^x)$ **2**

Question 14 (15 marks) Use a SEPARATE writing sheet.

a) On an Argand diagram let $A = 3 + 4i$

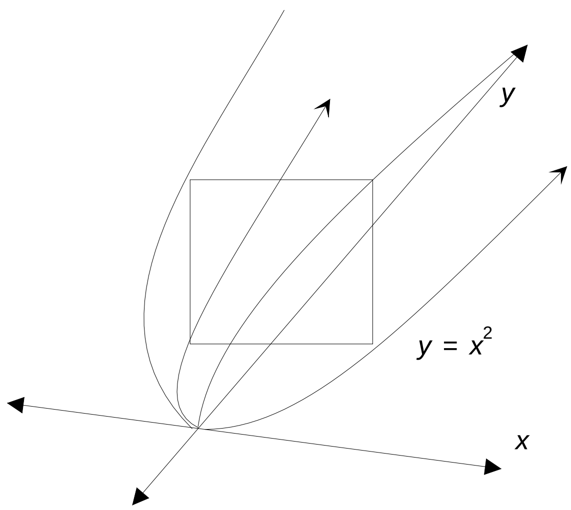
(i) Draw a clear sketch to show the important features of the curve defined by $|z - A| = 5$ **2**

(ii) Also for z on this curve, find the maximum value of $|z|$ **1**

b) (i) If $I_n = \int_0^\infty e^{-x} \sin^n x \, dx$, then prove that $I_n = \dfrac{n(n-1)}{n^2+1} I_{n-2}$ **3**

(ii) Hence evaluate $\int_0^\infty e^{-x} \sin^4 x \, dx$ **3**

c) Shown below is a solid which has as its base the parabola $y = x^2$ in the XY plane. Sections taken perpendicular to the Y axis are squares. The length of the figure is 9 units.

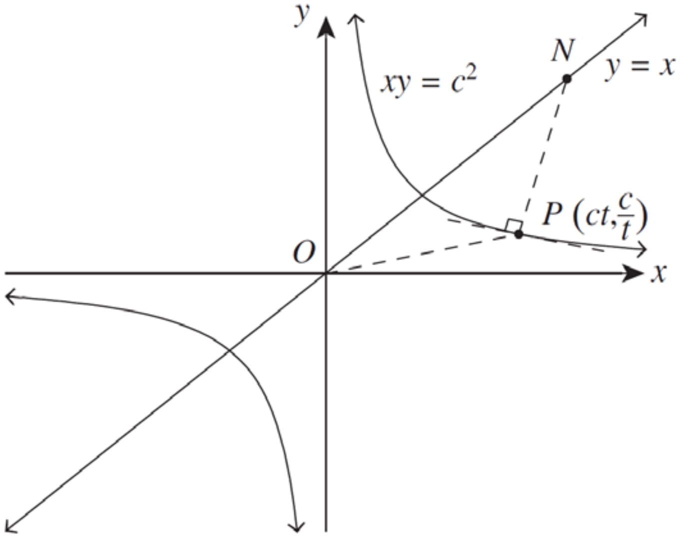


By using the technique of slicing, determine the volume of this solid. **3**

d) If $0 < x < y < \dfrac{1}{2}$ prove that: $\sqrt{xy} < x + y < \sqrt{x+y}$ **3**

Question 15 (15 marks) Use a SEPARATE writing sheet.

a) The diagram below shows the hyperbola $xy = c^2$. The point $P\left(ct, \frac{c}{t}\right)$ lies on the curve where $t \neq 0$. The normal at P intersects the straight line $y = x$ at N, O is the origin.



(i) Prove that the equation of the normal at P is $y = t^2 x + \frac{c}{t} - ct^3$ **2**

(ii) Find the coordinates of N **1**

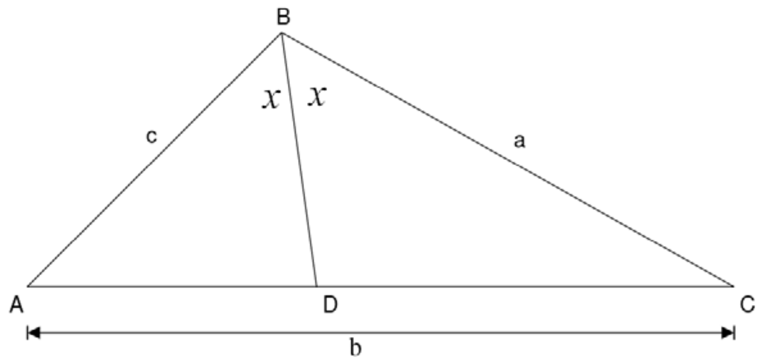
(iii) Show that triangle OPN is isosceles **2**

b) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts the x-axis at the points M and N .
A tangent drawn to the ellipse at a point $P(a\cos t, b\sin t)$ meets the tangents at M and N at the points Q and R respectively.

Given that the equation of the tangent at P is $\frac{x\cos t}{a} + \frac{y\sin t}{b} = 1$ (DO NOT PROVE THIS)

Prove that, for all positions of P, $MQ \times NR$ is a constant **2**

c) Evaluate $\int_0^1 \sin^{-1} x \, dx$ **3** d)



In ΔABC, BD bisects $\angle ABC$ as shown in the diagram.

i) By considering the area of ΔABC, show that **2**

$$BD = \frac{2ac\cos x}{a+c}$$

ii) Show that: $\cos x = \frac{1}{2}\sqrt{\frac{(a+c)^2 - b^2}{ac}}$ **2**

iii) Hence show that $BD = \frac{\sqrt{ac}}{a+c}\sqrt{(a+c)^2 - b^2}$ **1**

Question 16 (15 marks) Use a SEPARATE writing sheet.

a) Scientists use a pressure gauge which measures depth as it sinks towards the ocean floor. The gauge of mass 2 kg is released from rest at the ocean's surface. As it sinks in a vertical line, the water exerts a resistance to its motion of $4v$ Newtons, where $v\ \text{ms}^{-1}$ is the velocity of the gauge.

Let x be the displacement of the ball measured vertically downwards from the ocean's surface, t be the time in seconds elapsed after the gauge is released, and g be the constant acceleration due to gravity.

(i) Show that $\dfrac{d^2x}{dt^2} = g - 2v$ **2**

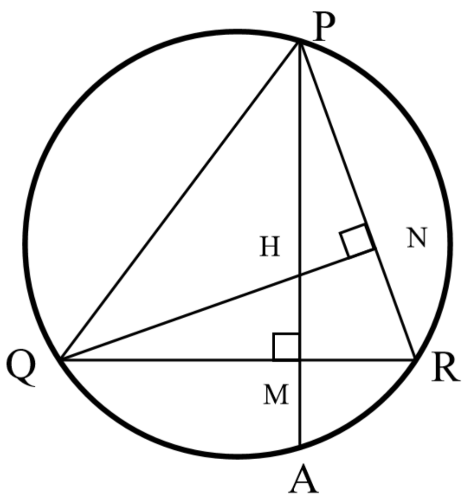
(ii) Hence show that $t = \dfrac{1}{2}\log_e\left(\dfrac{g}{g-2v}\right)$ **3**

(iii) Show that $v = \dfrac{g}{2}\left(1 - e^{-2t}\right)$ **2**

(iv) Write down the limiting (terminal) velocity of the gauge **1**

(v) At a particular location, the gauge takes 180 seconds to hit the ocean floor. Using $g = 10\ \text{ms}^{-1}$, calculate the depth of the ocean at that location, giving your answer correct to the nearest metre. **3**

b) The altitudes PM and QN of an acute angled triangle PQR meet at H. PM produced cuts the circle PQR at A. Copy the diagram onto your answer sheet.



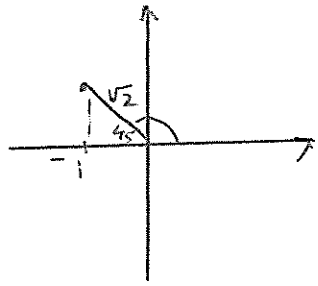
i) Explain why PQMN is a cyclic quadrilateral **1**

ii) Hence or otherwise prove that HM = MA. **3**

End of Exam

Multiple Choice

Q1

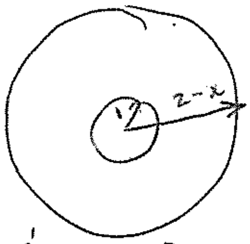


$\boxed{B}$

Q2 $\boxed{D}$

Q3 $\boxed{A}$

Q4



$V = \int_{-1}^{1} \pi \left[R^2 - r^2\right] dy$

$= \int_{-1}^{1} \pi \left[(2-x)^2 - 1^2\right] dy$

$= \int_{-1}^{1} \pi \left[(2+y^2)^2 - 1^2\right] dy$

$= \int_{-1}^{1} \pi \left[4 + 4y^2 + y^4 - 1\right] dy$

$= \int_{-1}^{1} \pi \left(y^4 + 4y^2 + 3\right) dy$

$\boxed{B}$

Q5 $a^2 = b^2(e^2-1)$

$a^2 = 12 \quad b^2 = 8$

$a = 2\sqrt{3} \quad b = 2\sqrt{2}$

$12 = 8(e^2-1)$

$\frac{12}{8} = e^2 - 1$

$e^2 = \frac{20}{8}$

$e = \frac{2\sqrt{5}}{2\sqrt{2}} = \frac{\sqrt{5}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{10}}{2}$

Foci $(0, \pm be)$

$\left(0, \pm 2\sqrt{2} \times \frac{\sqrt{10}}{2}\right)$

$= (0, \pm\sqrt{20})$

$= (0, \pm 2\sqrt{5})$

$\boxed{C}$

Q6 $\alpha = 1+\sqrt{2}i \quad \beta = 1-\sqrt{2}i \quad \gamma = ?$

$\alpha+\beta = 2$

$\alpha\beta = 3$

$\alpha+\beta+\gamma = 0 = -\frac{b}{a}$ —①

$\alpha\beta\gamma = -\frac{d}{a} = -q$ —②

$\alpha\beta+\beta\gamma+\alpha\gamma = +p$ —③

① $2+\gamma = 0$

$\gamma = -2$

③ $\alpha\beta\gamma = 3(-2) = -q$

$q = 6$

② $3 + -2(1-\sqrt{2}i) + -2(1+\sqrt{2}i) = +p$

$3 - 2 + \cancel{2\sqrt{2}i} - 2 - \cancel{2\sqrt{2}i} = +p$

$3 - 4 = +p$

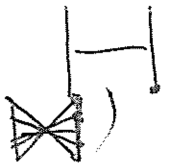
$p = 1$

$\boxed{B}$

Q7. All combinations

${}^{12}C_3$ -- (12 points choosing 3 vertices)

$3\,{}^{5}C_3$ — (3 ~~lines~~ sets of lines containing collinear points



4 — ( )

$\boxed{B}$

Q8 $\boxed{A}$

Q9 $\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx.$

$= x\tan x - \int \tan x \, dx$

$= \left[x\tan x + \ln \cos x \right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} + \ln\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2.$

$\boxed{A}$

Q10. $v \frac{dv}{dx} = \frac{1}{x^3}(6 - 10x)$

$v\,dv = \frac{1}{x^3}(6 - 10x)\,dx$

From $v = 0, x = 1$ to end $v = v$ $x = x$

$\int_0^v v\,dv = \int_1^x \left[\frac{1}{x^3}(6 - 10x)\right] dx$

$\left[\frac{v^2}{2}\right]_0^v = \left[\frac{-3}{x^2} + \frac{10}{x}\right]_1^x$

$\frac{v^2}{2} = -\frac{3}{x^2} + \frac{10}{x} + 3 - 10.$

$\frac{v^2}{2} = \frac{1}{x^2}(-3 + 10x - 7x^2)$

$v^2 = \frac{1}{x^2} \cdot 2(-3 + 10x - 7x^2)$

$v = \pm \frac{1}{x}\sqrt{2(-3 + 10x - 7x^2}$

$\boxed{C}$

Question 11.

a) (i) $\frac{z}{w} = \frac{3-i}{2+i} \times \frac{2-i}{2-i}$

$= \frac{6-3i-2i+i^2}{4-i^2}$

$= \frac{5-5i}{5}$

$= 1-i$

(ii) $\overline{-2iz} = \overline{-2i(3-i)}$

$= \overline{-2(3i-i^2)}$

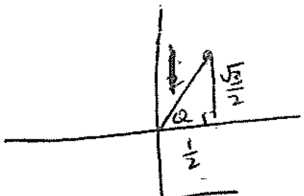
$= \overline{-2(3i+1)}$

$= \overline{-6i-2}$

$= -2+6i$

b) $z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

(i)



Mod $= \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$= 1$

Arg $= \tan^{-1}(\sqrt{3})$

$= \frac{\pi}{3}$.

$z = \text{cis}\frac{\pi}{3}$.

(ii) $z^6 = \left(\text{cis}\frac{\pi}{3}\right)^6$

$= (\text{cis } 2\pi)$ by De Moivre's Theorem

$= 1$.

(iii) $z^6 - 1 = 0$

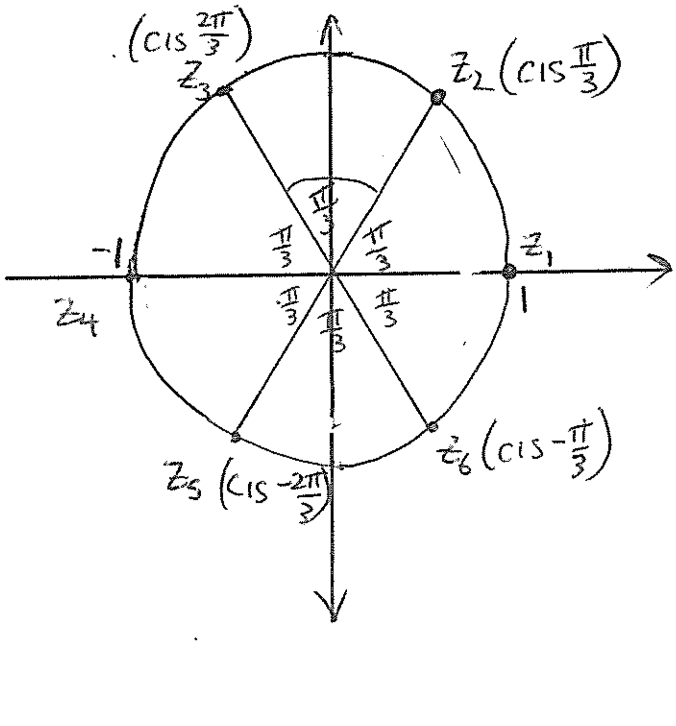
$z^6 = 1 = \text{cis}(2k\pi)$, k integer

$z = \text{cis}\left(\frac{2k\pi}{6}\right)$

$= \text{cis}\left(\frac{k\pi}{3}\right)$ $k = 0, \pm 1, \pm 2, -3$

$z = 1, -1, \text{cis}\frac{\pi}{3}, \text{cis}\left(-\frac{\pi}{3}\right),$

$\text{cis}\left(\frac{2\pi}{3}\right), \text{cis}\left(-\frac{2\pi}{3}\right)$



c) $\int \frac{dx}{x\sqrt{x^2-1}}$ $\quad x = \sec\theta$

$dx = \sec\theta \tan\theta \, d\theta.$

$$\int \frac{\sec\theta . \tan\theta \, d\theta}{\sec\theta . \tan\theta} = \int d\theta$$

$$= \theta + C$$

$$= \sec^{-1} x + C$$

d) $\int_0^1 \frac{\sqrt{x}}{1+x} dx = 2 - \frac{\pi}{2}$

Let $u = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$

$x=0 \quad u=0$

$x=1 \quad u=1.$

$du = \frac{1}{2u} dx$

$$= \int_0^1 \frac{u . 2u \, du}{1+u^2}$$

$$= 2\int \frac{(1+u^2-1)}{1+u^2} du$$

$$= 2\int_0^1 du - 2\int_0^1 \frac{1}{1+u^2} du$$

$$= 2\left[u\right]_0^1 - 2\left[\tan^{-1} u\right]_0^1$$

$$= 2 - 2 \times \frac{\pi}{4}$$

$$= 2 - \frac{\pi}{2}$$

Question 12

a (i) $y = \frac{1}{x^2}$

$y = 4x+8$

for intersection points A, B, C

$4x+8 = \frac{1}{x^2}$

$4x^3 + 8x^2 - 1 = 0$

(ii) $y = x^2$

$x = y^{\frac{1}{2}}$

$4(y^{\frac{1}{2}})^3 + 8(y^{\frac{1}{2}})^2 - 1 = 0$

$4y^{\frac{3}{2}} + 8y - 1 = 0$

$4y^{\frac{3}{2}} = 1 - 8y$

$16y^3 = 1 + 64y^2 - 16y$

$16y^3 - 64y^2 + 16y - 1 = 0$

since y is a dummy variable

$16x^3 - 64x^2 + 16x - 1 = 0$

(iii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2}{\alpha^2\beta^2\gamma^2}$

$= \frac{\sum 2 \text{ at a time}}{\text{product of roots}}$

$= \frac{1}{\frac{1}{16}}$

$= 16.$

(iv) $OA^2 = \alpha^2 + \left(\frac{1}{\alpha^2}\right)^2 = \alpha^2 + \frac{1}{\alpha^4}$

$OB^2 = \beta^2 + \frac{1}{\beta^4}$

$OC^2 = \gamma^2 + \frac{1}{\gamma^4}$

$\therefore OA^2 + OB^2 + OC^2 = \alpha^2 + \beta^2 + \gamma^2 + \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4}$

Now $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} = \frac{\alpha^4\beta^4 + \alpha^4\gamma^4 + \beta^4\gamma^4}{\alpha^4\beta^4\gamma^4}$

and $\alpha^4\beta^4 + \alpha^4\gamma^4 + \beta^4\gamma^4$

$= (\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2)^2 - 2\alpha^2\beta^2\gamma^2(\alpha^2 + \beta^2 + \gamma^2)$

$= 1^2 - 2 \times \frac{1}{16} \times 4$

$= \frac{1}{2}.$

$\alpha^4\beta^4\gamma^4 = \frac{1}{256}.$

$\therefore \alpha^2 + \beta^2 + \gamma^2 + \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4}$

$= 4 + \frac{256}{2}.$

$= 132.$

b) (i) $z^3=1$ $\quad w$ complex root.

$z^3-1=0$

$(z-1)(z^2+z+1)=0$

as $w\neq 0$ $\quad w$ root.

$\therefore w^2+w+1=0$

(ii) $(1+2w+3w^2)(1+2w^2+3w)$

$=[1+2w+3(-1-w)][1+2(-1-w)+3w]$

$=(1+2w-3-3w)(1-2-2w+3w)$

$=(-2-w)(-1+w)$

$=2-2w+w-w^2$

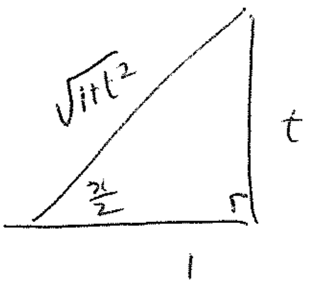
$=2-w-w^2$

$=2+1 \quad (1+w+w^2=0)$

$=3$.

c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{5+3\cos x}\,dx$

Let $t=\tan\frac{x}{2}$



$\frac{dt}{dx}=\sec^2\frac{x}{2}\cdot\frac{1}{2}$

$\frac{dt}{dx}=\frac{1}{2}(1+t^2)$

$\frac{2\,dt}{1+t^2}=dx$ $\qquad x=0,\ t=0$
$\qquad\qquad\qquad x=\frac{\pi}{2},\ t=1$

$$I=\int_0^1\frac{2\,dt}{(1+t^2)\left(5+3\times\frac{1-t^2}{1+t^2}\right)}$$

$$=\int_0^1\frac{2\,dt}{5+5t^2+3-3t^2}$$

$$=\int_0^1\frac{2\,dt}{8+2t^2}$$

$$=\int_0^1\frac{dt}{4+t^2}$$

$$=\frac{1}{2}\left[\tan^{-1}\frac{t}{2}\right]_0^1$$

$$=\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$$

Question 13

a) $\displaystyle\int \frac{dx}{\sqrt{9-9+6x-x^2}}$

$= \displaystyle\int \frac{dx}{\sqrt{9-(x-3)^2}}$

$= \sin^{-1}\left(\frac{x-3}{3}\right) + C$

b(i) $\dfrac{x+4}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$

$x+4 = A(x^2+4) + (Bx+C)x$

$x+4 = (A+B)x^2 + Cx + 4A.$

$4A = 4 \Rightarrow \boxed{A=1}$

$Cx = x \Rightarrow \boxed{C=1}$

$A+B=0 \Rightarrow \boxed{B=-1}$

(ii) $\displaystyle\int \frac{x+4}{x(x^2+4)}\,dx$

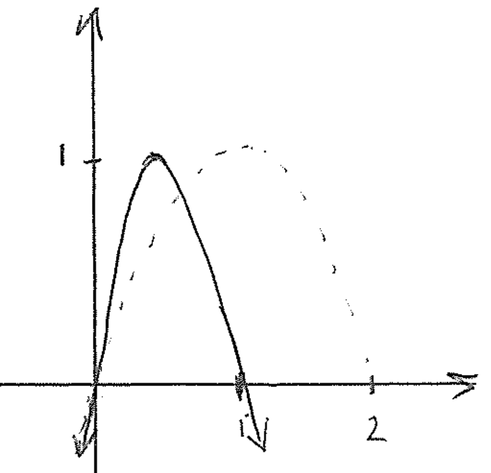
$= \displaystyle\int \frac{1}{x}\,dx + \int \frac{-x+1}{x^2+4}\,dx$

$= \displaystyle\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+4}\,dx + \int \frac{dx}{x^2+4}$

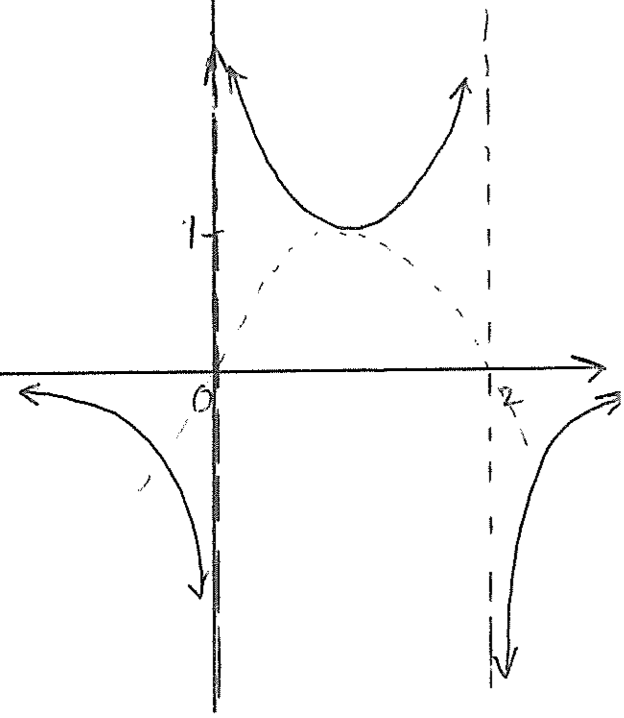
$= \ln(x) - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + C$

$= \ln\left(\dfrac{x}{\sqrt{x^2+4}}\right) + \frac{1}{2}\tan^{-1}\frac{x}{2} + C$

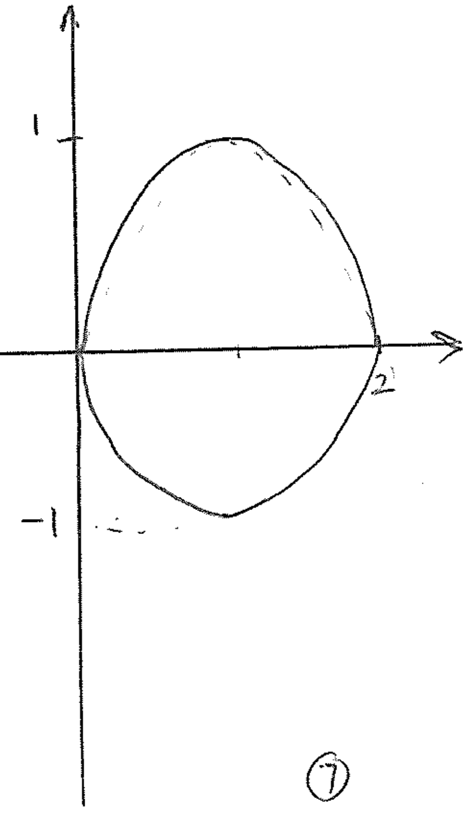
c(i)



(ii)



(iii)



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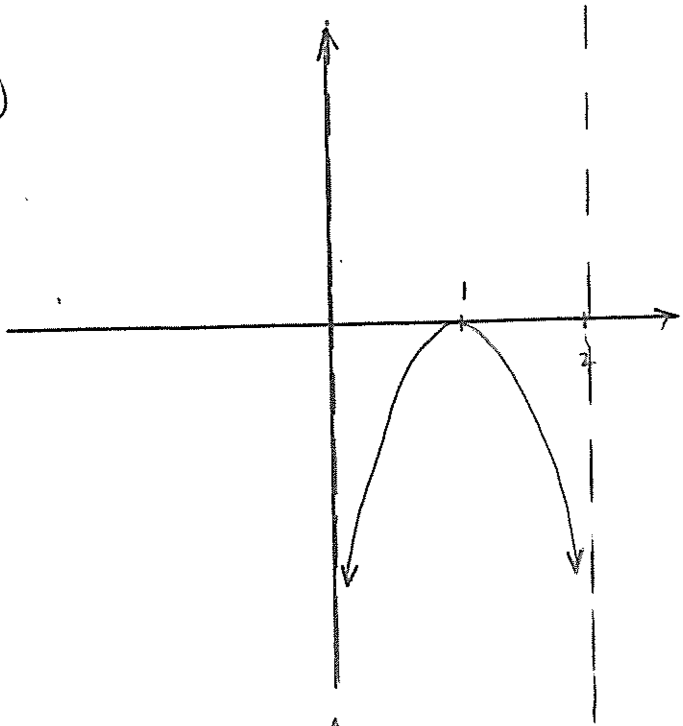
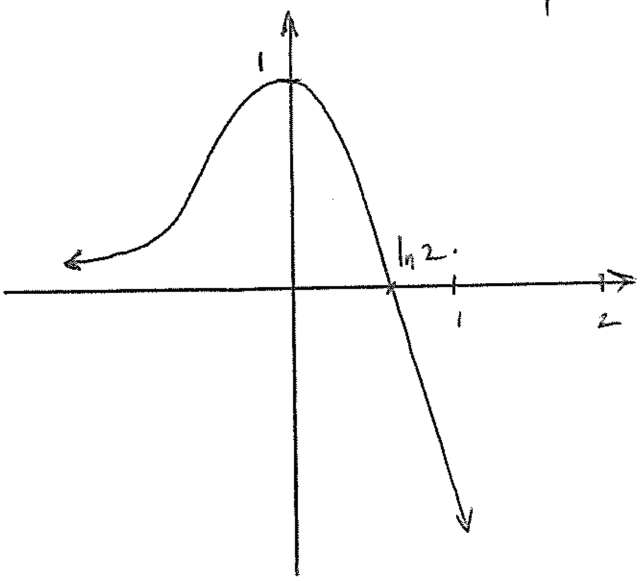
(iv)



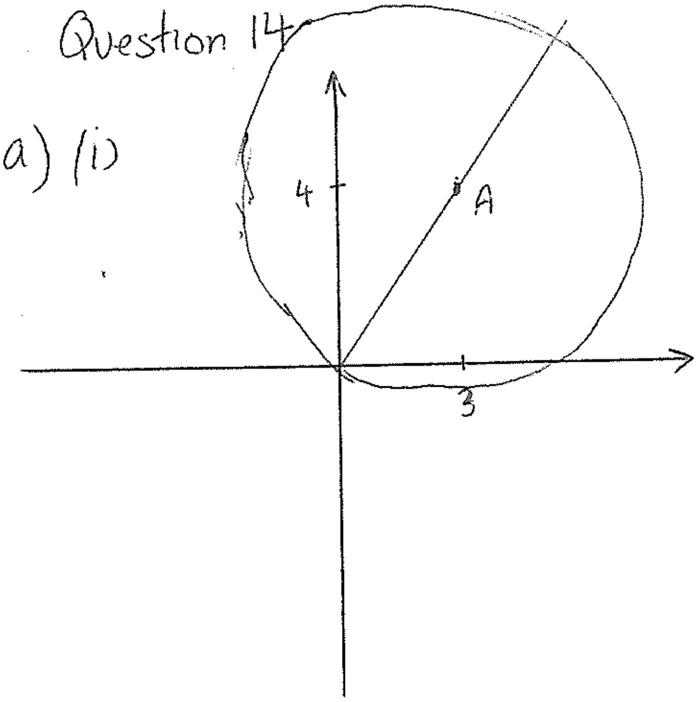
(V)



⑧

Question 14

a) (i)



(ii) max $|z|$ line from origin to other side of the circle

max $|z| = 10$ as $|OA| = 5$ units

b)(i) $I_n = \int_0^\infty e^{-x} \sin^n x \, dx$

$= \left[\sin^n x \, \frac{e^{-x}}{-1}\right]_0^\infty - \int_0^\infty n \sin^{n-1} x . \cos x . \frac{e^{-x}}{-1} \, dx$

(integration by parts)

$= n \int_0^\infty (\sin^{n-1} x \cos x) e^{-x} \, dx$ $(e^{-\infty} = 0)$

$= n \left[\left[\sin^{n-1} x \cos x \frac{e^{-x}}{-1}\right]_0^\infty - \int_0^\infty \left[(n-1)\sin^{n-2} x \cos^2 x - \sin^{n-2} x . \sin x\right] \frac{e^{-x}}{-1} dx \right]$

$= n(n-1) \int_0^\infty \sin^{n-2} x (1 - \sin^2 x) e^{-x} dx - n \int_0^\infty \sin^n x . e^{-x} dx$

$I_n = n(n-1) I_{n-2} - n(n-1) I_n - n I_n$

$I_n + (n^2 - n + n) I_n = n(n-1) I_{n-2}$

$(n^2+1) I_n = n(n-1) I_{n-2}$

$I_n = \frac{n(n-1)}{n^2+1} I_{n-2}$

(ii) $\int_0^\infty e^{-x} \sin^4 x \, dx = I_4$

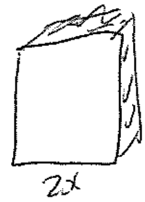
$I_4 = \frac{4(4-1)}{4^2+1} I_2$

$I_2 = \frac{2(2-1)}{2^2+1} I_0$

$= \frac{2}{5} \times 1 = \frac{2}{5}$

$I_4 = \frac{4 \times 3}{17} \times \frac{2}{5} = \frac{24}{85}$

c) $\delta V = \sum_{\delta y} (2x)^2 \, \delta y$



$V = \int_0^9 4x^2 dy$

$= \int_0^9 4y \, dy$

$= \left[\frac{4y^2}{2}\right]_0^9$

$= 162 \, u^3$.

d) $(\sqrt{x} - \sqrt{y})^2 \geqslant 0$

$x + y - 2\sqrt{xy} \geqslant 0$

$x + y > 2\sqrt{xy}$

$\therefore x + y > \sqrt{xy}$ ---- ①

If $x \leq \frac{1}{2}$ and $y \leq \frac{1}{2}$ then

$x + y \leq 1$

$\therefore \sqrt{x+y} \geqslant x + y$ — ②

as the square root of a number between 0 and 1 is greater than the number.

$\therefore$ from ① and ②

$\sqrt{xy} \leq x + y \leq \sqrt{x+y}$

Question 15

a) $xy = c^2$

$xy' + y = 0$

$y' = \frac{-y}{x}$

$= \frac{-c}{t \times ct} = -\frac{1}{t^2}$

$\therefore$ m of the normal at $P = t^2$

equation of the normal

$y - \frac{c}{t} = t^2(x - ct)$

$y = t^2 x + \frac{c}{t} - ct^2$

(ii) for N, $y = x$

$x = t^2 x + \frac{c}{t} - ct^2$

$x(1 - t^2) = \frac{c}{t}(1 - t^4)$

$x = \frac{c(1+t^2)(1-t^2)}{t(1-t^2)}$

$x = \frac{c(1+t^2)}{t}$

$y = \frac{c(1+t^2)}{t}$

(iii) $|OP| = \sqrt{(ct)^2 + \left(\frac{c}{t}\right)^2}$

$= \frac{c}{t}\sqrt{t^4+1}$

$$\boxed{|PN| = \sqrt{\left(ct - \frac{c(1+t^2)}{t}\right)^2 + \left(\frac{c}{t} - \frac{c(1+t^2)}{t}\right)^2}}$$

$= \frac{c}{t}\sqrt{(t^2 - 1 - t^2)^2 + (1 - 1 - t^2)^2}$

$= \frac{c}{t}\sqrt{1+t^4}$ $\therefore |OP| = |PN|$

ΔOPN is an isosceles triangle

b) since the tangent at M and N are vertical lines

$\therefore$ x-coordinates of Q and R are $a, -a$.

$Q_y = (1 - \cos t)\frac{b}{\sin t}$

$R_y = (1 + \cos t)\frac{b}{\sin t}$.

$MQ \times NR = (1-\cos t)\frac{b}{\sin t} \times (1+\cos t)\frac{b}{\sin t}$

$= (1 - \cos^2 t)\frac{b^2}{\sin^2 t}$

$= b^2$ (which is a constant)

c) $\int_0^1 1 . \sin^{-1} x \, dx$

$= \left[\sin^{-1} x . x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$

$= \frac{\pi}{2} - \frac{1}{2}\int_0^1 \frac{2x}{\sqrt{1-x^2}} dx$

$u = 1 - x^2$

$\frac{du}{dx} = -2x$

$du = -2x \, dx$

$= \frac{\pi}{2} + \frac{1}{2}\left[\frac{u^{-\frac{1}{2}+1}}{\frac{1}{2}}\right]$

$= \frac{\pi}{2} + \left[\sqrt{1-x^2}\right]_0^1$

$= \frac{\pi}{2} + 0 - 1$

$= \frac{\pi}{2} - 1$.

10

(i) Area of $\Delta ABC = \frac{1}{2}ac\sin 2x$

$$= \frac{1}{2}ac \times 2\sin x\cos x$$

$$= ac\sin x\cos x$$

Area of $\Delta ABD = \frac{1}{2} c \times BD \sin x$

Area of $\Delta BDC = \frac{1}{2} BD \times a \sin x$

$\therefore ac\sin x\cos x = \frac{1}{2}BD(c\sin x + a\sin x)$

$$\frac{2ac\cos x}{a+c} = BD$$

(ii) $$\cos 2x = \frac{a^2+c^2-b^2}{2ac}$$

$$2\cos^2 x - 1 = \frac{a^2+c^2-b^2}{2ac}$$

$$2\cos^2 x = \frac{a^2+c^2-b^2}{2ac} + 1$$

$$= \frac{(a+c)^2-b^2}{2ac}$$

$$\cos^2 x = \frac{1}{4}\left[(a+c)^2-b^2\right]$$

$$\cos x = \frac{1}{2}\sqrt{\frac{(a+c)^2-b^2}{ac}}$$

Hence $$BD = \frac{2ac}{a+c} \times \frac{1}{2}\sqrt{\frac{(a+c)^2-b^2}{ac}}$$

$$BD = \frac{\sqrt{ac}}{a+c}\sqrt{(a+c)^2-b^2}$$

$\downarrow mg \quad \uparrow 4v$

Question 16

a(i) $m\ddot{x} = mg - 4v$, $m = 2kg$

$\therefore \ddot{x} = g - 2v$

(ii) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dv}{dt} = v\frac{dv}{dt}$

$\frac{dv}{dt} = g - 2v$

$\int_0^v \frac{dv}{g-2v} = \int_0^t dt.$

$-\frac{1}{2}\left[\ln(g-2v)\right]_0^v = t.$

$\Rightarrow t = -\frac{1}{2}\left(\ln(g-2v) - \ln g\right)$

$t = \frac{1}{2}\ln\left(\frac{g}{g-2v}\right)$

(iii) $2t = \ln\frac{g}{g-2v}$ (from ii)

$e^{2t} = \frac{g}{g-2v}$

$g - 2v = ge^{-2t}$

$2v = g - ge^{-2t}$

$v = \frac{g}{2}(1 - e^{-2t})$

(iv) $\ddot{x} \to 0 \Rightarrow g - 2v \to 0$

$\Rightarrow v \to \frac{g}{2}$

OR.

$t \to \infty,\ e^{-2t} \to 0 \Rightarrow v \to \frac{g}{2}$

(part iii)

(v) $\therefore V = \frac{g}{2}(1 - e^{-2t})$

$g = 10$

$V = 5(1 - e^{-2t})$

$\int dx = \int 5(1 - e^{-2t})dt.$

$x = 5\left[t - \frac{e^{-2t}}{-2}\right] + c.$

$= 5\left[t + \frac{e^{-2t}}{2}\right] + c$

$x = 0 \quad t = 0$

$0 = 5\left(\frac{1}{2}\right) + c$

$c = -\frac{5}{2}$

$x = 5\left(t + \frac{e^{-2t}}{2}\right) - \frac{5}{2}$

$t = 180 \quad x = ?$

$x = 5\left(180 + \frac{e^{-360}}{2}\right) - \frac{5}{2}$

$\doteqdot 898$ m (nearest metre)

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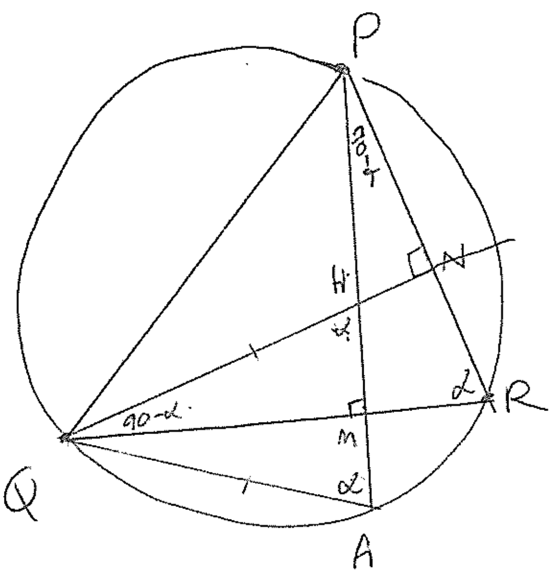
b(i)



PQMN is a cyclic quadrilateral
PQ subtends $90°$ at M and N
$\therefore$ PQMN is a cyclic quadrilateral with PQ as diameter

(ii) $HM = MA$

Let $\angle PRQ = \alpha$.

$\therefore \angle PRQ = \angle PAQ = \alpha$.

($\angle$'s on circumference subtended by arc PQ, circle PQR)

$\angle RPA = 90 - \alpha$ ($\angle$ sum of ΔPRM)

$\angle RPA = \angle NQR = 90 - \alpha$

($\angle$'s on circumference subtended by arc MN, cyclic PQMN)

$\therefore \angle QHM = \alpha$ ($\angle$ sum of ΔQHM)

$\therefore \angle QHM = \angle QAH = \alpha$

$\therefore HQ = QA$ ($\angle$'s opposite equal sides ΔQHA)

$\therefore HM = MA$.

($\Delta QMH \equiv \Delta QMA$

QM common

$\angle QHA = \angle QAH = \alpha$

$QH = QA$

(shown)

(13)